Overhangs in interface growth and ground-state paths

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We report on numerical investigations concerning the statistics of the ground-state path in a disordered medium. For a distribution of energies having a positive support, these paths are almost directed, even though some overhangs are present. Computing both the directed minimum energy path and the unconstrained ground-state paths allows us to understand whether the two problems belong to the same universality class. We present numerical results in dimensions up to $d=6=5+1$, which suggests that both problems are in the same universality class. [S1063-651X(98)01404-4]

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The Kardar-Parisi-Zhang (KPZ) equation [1] is probably the simplest nonlinear stochastic process. As such, it has become a cornerstone of statistical physics in the last decade. Its fame is also due to the concept of universality, which has led to grouping a large number of stochastic processes in the same KPZ universality class [2]. Universality is usually established via perturbative methods close to an upper critical dimensionality. It was soon realized, however, that with perturbative methods one cannot go very far since, in renormalization group jargon, the physics of the KPZ equation is described by a nonperturbative strong coupling fixed point. The paradigm of a KPZ universality, which is mostly based on numerical results in low dimensions, has recently been questioned. In particular, in Ref. [3] it was suggested that higher order nonlinear terms, usually ruled out on the basis of dimensional analysis, could be relevant at a strong coupling nonperturbative fixed point. Indeed evidence of nonuniversality was found in an infinite-dimensional limit of a modified KPZ equation [4]. More recently, nonuniversality in the change of the disorder distribution was discovered [5].

In this work, universality with respect to a different change in the model is questioned. From the point of view of interface growth, the KPZ equation can be derived [3] as a description of several ''microscopic'' models such as the Eden model [6] and restricted solid on solid (RSOS) models [7]. In the first model the interface has overhangs, whereas in the second these are forbidden. The irrelevance of overhangs has been proven numerically in $D=1+1$ dimensions. No such analysis has been carried out in higher dimensions. Clearly in high dimensions overhanging configurations are entropically favored. Therefore it is important to address this issue. It was first realized in Ref. [8] that interface growth models map into the problem of polymers in random media. Given a random medium, specified in terms of random ''energies'' e_x defined on each site $x=(x_1,\ldots,x_D)$ of a D-dimensional lattice, the problem consists of finding the path of lowest total energy between two points. The mapping is based on the fact that e_x can be interpreted as waiting times for the occupation of site x once at least one of its neighbors has been reached by the growing cluster. Then a given site A will be reached by the growth process starting at $x=0$, at a time T_A, which is the smallest sum of the waiting times along all possible paths from 0 to A. More precisely, let

$$E_p=\sum_{t=0}^{\ell} e_{x_p(t)}$$

be the energy of path $p=\{x_p(0)=0,\ldots,x_p(\ell)=A\}$. Then

$$T_A=\min_{p:x_p(0)=0\,x_p(\ell)=A} E_p\equiv E_g\,.$$

The problem is then to find the path g with the smallest energy E_g, i.e., the ground-state path (GSP). Clearly, since waiting times are positive, $e_x>0$ and the minimum will most likely be realized on the shortest possible paths, i.e. on directed paths. For RSOS growth, only directed paths are allowed, since overhangs are forbidden. This problem is than related to that of finding the path d with smallest energy E_d among all directed paths. It was shown [8] that for the Eden model the correct distribution of e_x is the exponential $\mathcal{P}(e)=\exp(-e)$. With similar arguments, one can show that polynuclear growth is equivalent to directed polymers with a bimodal distribution $\mathcal{P}(e)=p\,\delta(e-1)+(1-p)\,\delta(e)$. Nonuniversality with respect to the change of $\mathcal{P}(e)$ reported in Ref. [5] already suggests that these models do not belong to the same universality class. Here we wish to address the issue of universality with respect to the formation of overhangs. For this reason we shall fix $\mathcal{P}(e)$ once and for all. Since we found huge crossovers, even in $D=1+1$, for the exponential distribution, and bad statistics for the bimodal one, here we consider the uniform distribution $\mathcal{P}(e)=1$ for $e\in[0,1]$, which is also convenient from the numerical point of view.

The directed ground-state path was found via a standard transfer matrix method [2] in the $(1,1,\ldots,1)$ direction,

$$E(x,t+1)=\min_{\langle y,x\rangle,|y|=t}\,[E(y,t)]+e_x\,, \tag{1}$$

where $\langle y,x\rangle$ denotes the set of nearest neighbors of x, and the minimum is restricted to points at a longitudinal distance $|y|=t$ ($|y|=\Sigma_{i=1}^{D}y_i$ is defined as the sum of the components of y). The directed path to the point A is obtained by a usual trace back procedure [2] on the energy landscape $E(x,|x|)$. Equation (1) is iterated only on the hyperplane $|x|=t+1$.

A transfer matrix method also exists for the nondirected path. Starting from an energy configuration $E(x,0)=\infty$ for $x\neq 0$ and $E(0,0)=0$, and iterating the equation

$$E(x,t+1)=\min_{\langle y,x\rangle}[E(y,t)]+e_x \tag{2}$$

on all points x of the lattice, one clearly obtains the nondirected GS path. Equation (2) has a simple interpretation as a transfer matrix. Indeed if t is considered as a further $D+1$st ''time'' direction of the lattice, Eq. (2) can be considered as the transfer matrix in the ''time'' direction. In this augmented space, the problem corresponds to directed polymers in a columnar disorder [9,2]. Indeed, the disorder depends only on x and not on ''time.'' The GS energy is then obtained as $E_g(\ell)=\min_t E(A,t)$. A traceback procedure is also possible in this case. Instead of Eq. (2), however, one has to iterate

$$E(x,t+1)=\min\{\min_{\langle y,x\rangle}[E(y,t)]+e_x,E(x,t)\}, \tag{3}$$

so that at each site the energy $E(x,t)$ can only decrease with time. In particular, once the GS path from 0 to x has arrived, the energy $E(x,t)$ will not change anymore. So the iteration of Eq. (3) identifies the energy landscape of GS paths to each point x of the lattice. On this landscape it is easy to trace back [2] the GS path from 0 to A. Note that equal energy hypersurfaces on this landscape identify the surface of the corresponding growth model.

Equations (1) and (2) were iterated numerically on a hypercube of L^D sites. The origin $0=(0,0,\ldots,0)$ was chosen at one of the vertices of the lattice, whereas the point A was taken at the opposite vertex $A=(L-1,L-1,\ldots,L-1)$. The length of directed paths is $\ell=d*(L-1)$. Finally we considered closed boundary conditions in order to consider only paths which cross the entire lattice. The reason for the choice of this peculiar geometry is that in this way one avoids anysotropy effects [10] induced by the discrete lattice. The statistics is based on $N\simeq 2000$ realizations of the disorder. For each realization directed and nondirected paths were found.

The ground-state path g is, in general, not directed and its length $||g||$ will be larger than that of the directed one ($\|d\|=\ell$). Since every step taken in the negative direction needs an extra step in the positive one, we shall set $\|g\|=\ell+2n$, where n then counts the number of steps in negative directions. Clearly n cannot grow faster than ℓ, since this would require an average energy per site, on g, which is vanishingly small as $\ell\to\infty$. We indeed found that n/ℓ increases with D, and it ranges from 0.6% in $D=2$ to 6% in $D=6$ [see Fig. 2 (a)].

The GSP energies satisfy, in general, the scaling behaviors[2]

$$E_d(\ell)=\epsilon_d\ell+a_d\ell^{\omega_d}X_d, \tag{4}$$

$$E_g(\ell)=\epsilon_g\ell+a_g\ell^{\omega_g}X_g, \tag{5}$$

where ϵ_α and a_α ($\alpha=g,d$) are numerical coefficients, whereas X_α is a random variable which depends on the realization of the energy landscape. In other words, Eqs. (4) and (5) assume that the energies of ground-state paths have an extensive nonfluctuating part plus a subextensive fluctuation contribution. A second quantity one can consider is the displacement $\xi(t)=|x_p(t)|_\perp$ of the path in the transverse direction, where $|x|_\perp$ is the distance from x to the point $x^{(0)}=(|x|/D,|x|/D,\ldots,|x|/D)$. For $t\ll\ell$ one has $\xi(t)\sim t^{\nu_\alpha}$, where again $\alpha=d$ or g. The exponents ω_α and ν_α are usually expected to obey the relation $\omega_\alpha=2\nu_\alpha-1$, which follows from Galilean invariance in the noisy Burgers equation [11].

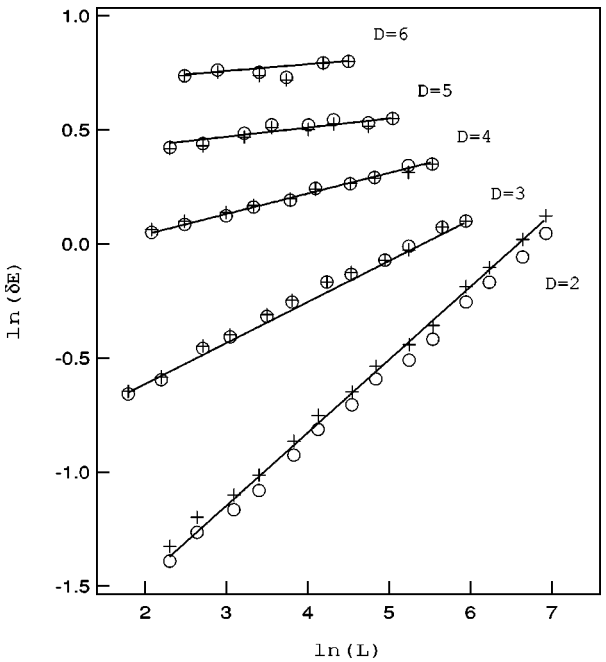

FIG. 1. Scaling of the fluctuations δE_d ($\bigcirc$) and δE_g ($+$) of the energy of the ground-state paths for dimensions $D=2,\ldots,6$.

Universality, in the present context would correspond to the statement $\omega_d=\omega_g$ and $\nu_d=\nu_g$. Double logarithmic plots of the fluctuation of the path energies are shown in Figure 1. Our numerical accuracy did not allow for a very precise evaluation of the exponents. They, however, clearly show no sign of deviation from universality. The values of the exponents $\omega(D)$ and $\nu(D)$ are listed in Table I.

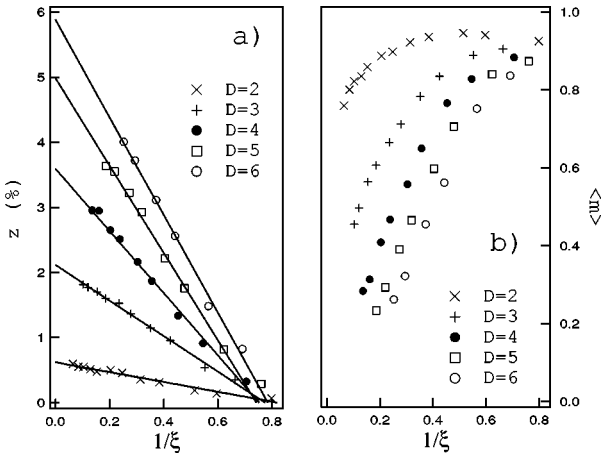

FIG. 2. (a) Percentage $z=n/\ell\times 100$ of nondirected steps on the ground-state path for different sizes ℓ and dimensions D. (b) Average overlap $\langle m\rangle$ between the paths d and g. Both z and $\langle m\rangle$ have been plotted vs $1/\xi(\ell)$ for convenience.

TABLE I. Energy fluctuation and wandering exponents as a function of D.

d	$L_{\max}$	ω_g	ω_d	$\nu_g \cong \nu_d$
2	2046	0.32(1)	0.33(1)	0.65(1)
3	381	0.18(1)	0.18(1)	0.59(1)
4	252	0.08(1)	0.09(1)	0.54(1)
5	155	0.04(2)	0.04(1)	0.51(1)
6	90	0.03(3)	0.03(2)	0.50(1)

It has to be pointed out, however, that finite size effects do occur. In order to address this point, we note that $E_g(\ell) \leqslant E_d(\ell)$, and from Eqs. (4) and (5) we expect that $E_d(\ell) - E_g(\ell) \propto \ell$. Each of these energies has fluctuations of order ℓ^{ω}. The fluctuations of E_g and E_d will clearly be correlated if $\delta E_g + \delta E_d > E_d - E_g$. This suggests the presence of finite size effects for scales

$$\ell < \ell_c = \left(\frac{a_d + a_g}{\epsilon_d - \epsilon_g} \right)^{1/(1-\omega)}.$$

Only for $\ell \gg \ell_c$ can the two paths possibly be statistically independent. Actually our numerical work could not access the regime $\ell \gg \ell_c$ due to resouces limitations. Our numerical results show that the correlation between the energies of the two paths is almost constant in $d=1$, and decreases in higher dimensions. For $d \geqslant 3$ it seems to follow the same curve.

The same effect also manifests itself in the overlap distribution $P_\ell(m)$ for paths of size ℓ. Here the overlap m is the fraction of sites which the directed path d shares with the nondirected path g. $P_\ell(m)$ shows a clear size dependence: for small ℓ it has a peak at $m=1$, whereas as ℓ increases it becomes broader and broader. Figure 2(b) shows the average overlap $\langle m \rangle$ vs $1/\xi$ for different values of ℓ and D. In dimensions $D>3$, $\langle m \rangle$ seems to converge to zero for large ξ (i.e., as $\ell \to \infty$).

In order to test the effect of correlation between g and d due to small size, we considered the scaling of δE and $\xi(t)$ considering only paths with overlap in an interval $[\mu, \mu + \Delta\mu]$. However, no systematic dependence of $\langle \delta E | m \in [\mu, \mu + \Delta\mu] \rangle$ was detected. This suggests that even in the absence of the spurious correlation arising from small sizes $\ell < \ell_c$, the scaling of the paths g and d remains the same, and it therefore supports universality. Even so, the possibility of nonuniversal behavior cannot be excluded for $\ell \gg \ell_c$.

[1] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[2] T. Halpin-Healey and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).

[3] M. Marsili, A. Maritan, F. Toigo, and J. R. Banavar, Rev. Mod. Phys. **68**, 963 (1996).

[4] M. Marsili and A. J. Bray, Phys. Rev. Lett. **76**, 2750 (1996).

[5] T. J. Newman and M. R. Swift, Phys. Rev. Lett. **79**, 2261 (1997).

[6] M. Eden, in *Symposium on Information Theory in Biology*, edited by H. P. Yockey (Pergamon, New York, 1958), p. 359.

[7] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).

[8] S. Roux, A. Hansen, and E. L. Hinrichsen, J. Phys. A **24**, L295 (1991).

[9] Y.-C. Zhang, Phys. Rev. Lett. **56**, 2113 (1986).

[10] D. Dhar, Phys. Lett. A **130**, 308 (1988).

[11] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).